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On the space energy distribution of slowed-down neutrons

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With 1 figure in the text

1. Introduction

The purpose of this paper is to study the distribution in space and energy of neutrons, slowed down by elastic collisions with the atomic nuclei of a moderator assumed to be homogeneous and infinitely large. The classical method of tackling this problem is to apply to the transport equation a Laplace transformation with respect to the logarithmic energy variable and a Fourier transformation with respect to the spatial coordinates. In the stationary and spherically symmetrical case, when in particular the source function is supposed to be a function only of the distance R from the centre of symmetry and of the energy E, one obtains for the number of collisions per unit volume, per unit energy range $F_0(R, E)$

$$F_0(R, E) = -\frac{1}{2\pi^2 R} \int_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta \, e^{x\eta} \int_{-\infty}^{+\infty} d\omega \, \omega \, e^{i\omega R} f_0(\omega, \eta). \tag{1}$$

Here the logarithmic energy variable x is defined as $x = \log \frac{E_0}{E}$, where E_0 is a constant. The path of integration in the η-plane is a straight line to the right of the singularities of the integrand.

Equation (1) is the starting point of our investigation. The general theory leading to this equation was first published by WALLER [8], [9]. Similar methods were also worked out in U.S.A. during the war, and appear in the comprehensive review by MARSHAK [6]. For details of the general theory we refer to these papers as well as a paper by VERDE and WICK [10]. We only point out here the underlying assumptions made in this theory which thus must be fulfilled in order that our basic equation (1) be valid. Inelastic scattering is neglected; the elastic scattering is assumed to be isotropic in the centre of gravity system of the neutron and the atom; the atoms of the medium are supposed to be at rest before a collision with a neutron occurs and their binding to other atoms are neglected. Besides these assumptions and those mentioned before, we also assume for simplicity that the neutrons are emitted from a point source, with an initial energy equal to E_0. Further we neglect absorption, though the way of treating a more general case without this restriction will also be shortly discussed.

In general, the computation of $F_0(R, E)$ consists of two problems; firstly the calculation of the function $f_0(\omega, \eta)$ in (1) and secondly the evaluation of the integral. However, in the general case when the mean free path λ of the neutrons is supposed to be a function of the energy E it is difficult to calculate $f_0(\omega, \eta)$ for all values of the variables necessary for the evaluation of (1) for any value of, for example, R. For small values of R (and of E) $F_0(R, E)$ can be easily computed even in the general case; this formula of $F_0(R, E)$ constitutes the well-known age approximation.[1] However, we intend to calculate the neutron density for any value of R, with the only restriction on the values of the parameters being imposed on E (or x), and, for simplicity, we shall therefore in this paper consider only the case when λ is constant. Then, for every moderator R is measured with λ as unity of length. The assumption of a constant λ within the slowing-down energy interval is fulfilled with good approximation only for sufficiently low values of E_0, roughly of the order 0.1 Mev. For higher E_0 we may try to take some mean value for λ. If this mean value is suitably chosen, the error introduced by this approximation is considerably reduced and we may expect a qualitative agreement, at least.

It is then easy to get an expression for $f_0(\omega, \eta)$, and we shall use that derived by WALLER [8], [9], having the form of a continued fraction

$$f_0(\omega, \eta) = \cfrac{q}{h_0(\eta) + \cfrac{\omega^2/3}{h_1(\eta) + \cfrac{4\,\omega^2/15}{h_2(\eta) + \cfrac{9\,\omega^2/35}{h_3(\eta) + \cdots}}}}, \tag{2}$$

with

$$q = \frac{Q}{4\pi E_0}, \tag{3}$$

where Q is the number of neutrons emitted per unit time [9].

Further

$$h_n(\eta) = 1 - c_n(\eta) \qquad (n = 0, 1, 2, \ldots). \tag{4}$$

The expression of $c_n(\eta)$ has also been derived by WALLER [8], [9]. Making a simple transformation in his formula we get[2]

$$c_n(\eta) = \frac{(M+1)^2}{2M} \int_{\delta}^{+1} P_n(\mu) \frac{[G(\mu)]^{2\eta}}{\sqrt{M^2 - 1 + \mu^2}}\, d\mu \qquad \delta = \begin{cases} 0; & \text{for } M = 1 \\ -1; & \text{for } M > 1 \end{cases}, \tag{5}$$

with

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[1] See for instance MARSHAK [6].

[2] Cf also WICK [11].

$$G(\mu)=\frac{\mu+(\mu^2+M^2-1)^{\frac{1}{2}}}{M+1}. \tag{6}$$

Here M is the mass of the scattering atomic nuclei measured in units of the neutron mass, $P_n(\mu)$ the n:th Legendre polynomial. Thus we have, for simplicity, assumed the moderator to contain one single slowing down element, the generalization to the case of a mixture is, however, very easily done [9]. Finally from the value of the collision number $F_0(R, E)$ we get the simple expression of the neutron density $N_0(R, E)=v^{-1}F_0(R, E)$ ($v=$ neutron velocity).

Before we continue with this investigation we shall briefly sum up some earlier calculations of the neutron density beyond the range of validity of the age approximation. WALLER [9] was the first to make such a calculation based on the general theory referred to above. He introduced the saddle point method in this problem. MARSHAK [6] proposed a method of successive approximations, giving a series development for the neutron density. This series, however, diverges at distances essentially beyond the range of validity of the age theory. PLACZEK[1] has made a partial summation of this series in order to use it at larger distances. However, the term he obtained is contained in WALLER's formula. In a paper by the present author [4] WALLER's method was used and extended to higher approximations. Important progress has been made by WICK [11]. He has used the saddle-point method and derived the asymptotic formula of the neutron density for large distances. He has also described a numerical method of calculating the neutron density at any distance. In this paper we shall develop a method making it possible, by means of analytic continuation, to obtain analytic formulas of the neutron density at all distances.

2. Performance of one integration in (1)

We first summarize some facts, well-known from the papers cited above, concerning the function $f_0(\omega, \eta)$. In an ω-plane cut along the imaginary axis from $-i\infty$ to $-i$ and from $+i$ to $+i\infty$ f_0 is a one-valued analytic function of ω with no other singularities than isolated poles provided η is chosen so that the relation

$$\lim_{n\to\infty} c_n(\eta)=0, \text{ i.e. } \lim_{n\to\infty} h_n(\eta)=1, \tag{7}$$

holds. This is always valid in the right halfplane of η. In the ω-plane f_0 has poles for certain values of ω, occurring, of course, in pairs with opposite signs and being in general functions of η. These poles are intimately connected with the zeroes of the functions $h_n(\eta)$ so that small values of the poles ω always correspond to η-values near to such a zero. One of the two poles corresponding to the zero with the largest real part, the root $\eta=1$, of the equation $h_0(\eta)=0$, gives the largest contribution to the integral (1) and is called the "main" pole ω_0.

Now, for $R\{\eta\}\to\infty$[2] which implies $h_n(\eta)\to 1$, f_0 has a very simple limiting function [11] regular in all the cut ω-plane and becoming infinite at $\omega=\pm i$.

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[1] See MARSHAK [6].

[2] $R\{\eta\}$ = the real part of η.

On the other hand, for real and finite values of $\eta > 1$ which will be seen later to be the most important in the evaluation of (1), all functions h_n are real, $h_n > 0$, and also $h_n < 1$ at least for the first few values of n. One may therefore guess, and it is also easily shown [11] that f_0 then has a purely imaginary pole ω in the interval 0, $+i$, which is in fact the pole called above the main pole ω_0.

It is always possible to perform one of the integrations in (1) by the method of residues. Following WALLER [9], we integrate ω first, using this method and are then left with an integration in η. The reverse order of integration can also be used [6], [11]. It is then convenient to introduce a variable θ instead of η,

$$\theta = \eta - 1. \tag{8}$$

Then, for the main pole $\omega_0 = \omega_0(\theta)$, we put

$$\omega_0(\theta) = i B(\theta). \tag{9}$$

For real positive θ-values $B(\theta)$ is always real and positive with $B(0) = 0$. From the preceding it is also obvious that

$$\lim_{\theta \to \infty} B(\theta) = 1. \tag{10}$$

A more detailed proof of this can be obtained by a direct study of $B(\theta)$ for large θ [11].

Concerning the other poles in the upper halfplane for real positive θ-values, they lie in general symmetrically around the imaginary axis with considerably larger imaginary parts than $\omega_0(\theta)$. For $\theta \to \infty$ they all move towards the cuts. In performing the integration in ω by closing the contour of integration by an infinite halfcircle in the upper half plane broken off by a path along the cut around the point $+i$ and forming the residues at the enclosed poles we see from (1) that the "secondary" poles will have very little importance compared to the main pole. In fact, we are going to evaluate the integral (1) only for such values of R and x that the secondary poles can be neglected.

We shall now consider more closely the main pole. In a small circle around the point $\theta = 0$ $B(\theta)$ may be determined by a series

$$[B(\theta)]^2 = C^2\,\theta\,(1 - b_1\,\theta - b_2\,\theta^2 - \cdots) = C^2\,\theta\,(1 - \beta_1\,\theta - \beta_2\,\theta^2 - \cdots)^2. \tag{11}$$

Here the constants C^2, b_1, $b_2 \ldots$, β_1, $\beta_2 \ldots$ can be successively determined. They will depend on the values of $h_n(\eta)$ and its derivatives at the point $\eta = 1$ with the notation

$$h_n^i = \frac{d^i}{d\,\eta^i}\{h_n(\eta)\} \tag{12}$$

taken for $\eta = 1$, ($i =$ symbol of differentiation, $n = 0, 1, 2, \ldots$). The expressions of some of the first constants in (11) in terms of h_n^i have been calculated before, obtaining [4]: $C^2 = 3\,h_0'\,h_1$ etc.

However, we must have accurate knowledge of $B(\theta)$ for much larger real θ than can be provided by (11). It can be obtained by analytic continuation of this expression of $B(\theta)$. In fact, taking, in the cut ω-plane and the θ-plane where (7) holds, those points ω, θ which make $|f_0^{-1}(\omega, \theta)|$ limited to some positive value $f_0^{-1}(\omega, \theta)$ is holomorphic both in ω and θ within these domains. Then, by the theorem of the permanence of the functional equation, the analytic continuation $\omega(\theta)$ of the expression (11) of $\omega_0(\theta)$ will always satisfy the equation $f_0^{-1}[\omega(\theta), \theta]=0$. In particular, the analytic continuation, along the real positive θ-axis and then along the central part of the path of integration, chosen sufficiently far from origin, always gives for $\omega(\theta)$ the main pole $\omega_0(\theta)$ as no branch points occur on these paths. This makes our definition of $B(\theta)$ more complete and also gives us a method of calculating it, facilitated for large θ by (10).

Calculating now the residue of f_0 at the main pole we obtain the value

$$\text{Res}\,\{f_0(\omega, \theta)\}_{\omega=\omega_0(\theta)} = \frac{q \cdot 3\,h_1}{2\,\omega_0(\theta)}\,A(\theta). \tag{13}$$

This defines a new function $A(\theta)$ for which we have $A(0)=1$, and in a small circle around this point

$$A(\theta)=1+\alpha_1\,\theta+\alpha_2\,\theta^2+\alpha_3\,\theta^3+\cdots. \tag{14}$$

Just as $B(\theta)$, $A(\theta)$ may be calculated outside this circle as the analytic continuation of the series (14). A direct study of $A(\theta)$ for large real θ shows that

$$\lim_{\theta\to\infty} A(\theta)=0 \tag{15}$$

which easily follows from a similar relation derived by WICK [11]. The constants α_ν may be successively calculated in terms of h_n^i. However, once the b_ν are calculated α_ν may be obtained directly with the use of a simple formula

$$\alpha_\nu = -\frac{\partial}{\partial(h_0')}[h_0'\,b_\nu] \qquad (\nu=1, 2, \ldots). \tag{16}$$

For the derivation of this formula see Appendix A. It was given in an earlier paper [4] but without proof; for explicit expressions of the three first α_ν see this paper.

Finally we obtain as the result of the integration in ω

$$F_0(R, E) = \frac{Q \cdot 3\,h_1}{8\,\pi^2\,E\,R}\,\frac{1}{i}\int\limits_{L_\theta} d\,\theta\, e^{x\,\theta - R\,B(\theta)}\,A(\theta). \tag{17}$$

Here the path of integration L_θ is a straight line in the right half plane parallel to the imaginary axis. (17) is obviously not exact, but the validity of the approximations made will be checked later.

3. Determination of the functions $B(\theta)$ and $A(\theta)$

The method we intend to use for the calculation of $B(\theta)$ and $A(\theta)$ was sketched above; for small $|\theta|$ by means of (11) and (14) and for other θ needed by analytic continuation of these series along the real positive θ-axis.

In order to carry out this program in practice we must start with a thorough knowledge of $B(\theta)$ and $A(\theta)$ in the neighbourhood of origin, which can always be obtained by calculating sufficiently many coefficients in the developments (11) and (14). However, the derivation of explicit formulas of b_ν and α_ν in h^i_n and then substitution of the values of h^i_n is an extremely laborious procedure when ν is not small. Therefore we have determined two recurrent systems of equations for the calculation of b_ν and α_ν which were found to make the numerical work rather easy. β_ν is then calculated by means of (11).

We define a function $A_{n-1}(\theta)$ by the continued fraction contained in (2), after substituting for ω the value at the main pole

$$A_{n-1}(\theta) = h_{n-1}(\theta) + \frac{\frac{n^2}{4n^2-1}[\omega_0(\theta)]^2}{h_n(\theta)] + \cdots} \qquad (n=1, 2, 3, \ldots). \tag{18}$$

$A_{n-1}(\theta)$ is a regular function at least for sufficiently small values of $|\theta|$ for which we write, in expanding $h_{n-1}(\theta)$, $h_n(\theta) \ldots$ and inserting (11)

$$A_n(\theta) = h_n \sum_{\nu=0}^{\infty} a_{n\nu}\, \theta^\nu \qquad (a_{n0}=1) \quad (n=1, 2, \ldots). \tag{19}$$

The recurrent system is obtained from the relation

$$[A_{n-1}(\theta) - h_{n-1}(\theta)]\, A_n(\theta) = -\frac{n^2}{4n^2-1}[B(\theta)]^2 \qquad (n=1, 2, \ldots), \tag{20}$$

written as an expansion for small $|\theta|$, observing that $A_0(\theta) \equiv 0$. The resulting system of equations for the $a_{n\nu}$ and b_ν is best written as two formulas

$$h_n\, h_{n-1} \sum_{\mu=1}^{\nu} \left(a_{n-1,\mu} - \frac{h^{\mu}_{n-1}}{\mu!\, h_{n-1}} \right) a_{n,\nu-\mu} = \frac{n^2}{4n^2-1} C^2\, b_{\nu-1} \tag{21}$$

$$(n=2, 3, \ldots, \ \nu=1, 2, \ldots),$$

with

$$b_0 = -1, \ C^2 = 3\, h'_0\, h_1,$$

and

$$b_{\nu-1} = -\sum_{\mu=1}^{\nu} \frac{h^{\mu}_0}{\mu!\, h'_0}\, a_{1,\nu-\mu} \qquad (\nu=1, 2, \ldots). \tag{22}$$

Using (16) we then obtain by differentiation of (21) and (22) another recurrent system for the determination of the α_ν. For abbreviation we put

$$d_{n\nu} = h_0' \frac{\partial}{\partial (h_0')} (a_{n\nu}), \tag{23}$$

and then we get

$$h_n h_{n-1} \sum_{\mu=1}^{\nu} \left[d_{n-1,\mu}\, a_{n,\nu-\mu} + \left(a_{n-1,\mu} - \frac{h_{n-1}^{\mu}}{\mu!\, h_{n-1}} \right) d_{n,\nu-\mu} \right] = \frac{-n^2}{4n^2-1} C^2 \alpha_{\nu-1} \quad (n=2,3,\ldots,\ \nu=1,2,\ldots), \tag{24}$$

with $\alpha_0 = 1$, $d_{n0} = 0$ $(n=1,2,\ldots)$,

$$\alpha_{\nu-1} = a_{1,\nu-1} + \sum_{\mu=1}^{\nu} \frac{h_0^{\mu}}{\mu!\, h_0'} d_{1,\nu-\mu} \qquad (\nu=1,2,\ldots). \tag{25}$$

If we now want to determine all b_ν and α_ν up to a certain index $\nu = k$, we start with a calculation of all h_n^i for which $n+i \leq k+1$ and then compute the corresponding $\frac{h_0^i}{i!\, h_0'}$ and $\frac{h_n^i}{i!\, h_n} (n \geq 1)$ and put them into a triangular scheme. Using (21) we then calculate $a_{n,1} - \frac{h_n'}{h_n} (n \leq k)$, and put them in the first column to the left of a second triangular scheme, and then $a_{n,1}$ and, remembering (22), b_1. After that we proceed to the second column of the scheme, containing $a_{n,2} - \frac{h_n''}{2\, h_n}$, $(n \leq k-1)$ and also get $a_{n,2}$ and b_2. We continue in this way until the value of b_k has been obtained. The intermediate values of $a_{n,\nu}$ are for every n value put in a row, beginning from the right. This makes the procedure easy in practice; in fact the calculation of a certain value $a_{n-1,\nu} - \frac{h_{n-1}^{\nu}}{\nu!\, h_{n-1}}$ consists according to (21) merely in applying the n:th row to the $n-1$ row of the scheme, forming the products with reversed signs of the contigous elements and adding the right member of (21). The calculation of such sums of products is very rapid using an electric computing machine.[1] The computation of the α_ν is then made in the same way, beginning with all $d_{n,1}$ and α_1, $d_{n,2}$ and α_2 etc. and, in order to facilitate the formation of the sums of products in (24), in putting $d_{n,\nu}$ in a triangular scheme similar to that used before, as well as for each n value in a row just as was made with the $a_{n,\nu}$ above.

The numerical calculations sketched above must be preceded by a computation of the constants h_n^i. Although exact explicit formulas may be derived for h_h^i, remembering (4) and $c_n(\eta)$ in the form given by WALLER [9] this point requires further attention. For $M=1$ this formula is certainly simple and h_n^i may be rapidly calculated in this way, but in the case $M>1$ a numerial computation is very involved, except for small values of n and i. The difficulties arise mainly from the fact that the exact expressions are small with increasing values of n and i but are given as algebraic sums of large terms, giving no account of the right order of magnitude of h_n^i for which, however, very simple relations exist. For instance for carbon, if $i \geq 4$ or $n \geq 2$ it is already difficult to get accurate values of h_n^i. Therefore, for $M > 1$ we have derived simple formulas suitable for numerical calculation (Appendix B). From these formulas it appears that for $M > 1$ we need not take into account more than the first few $c_n(\eta)$ in (2), putting all others $c_n = 0$. This fact has been pointed out before [6], but here we also may easily study in detail the validity of this approximation. Thus the numerical calculations show that for carbon $c_4(\eta)$ and for deuterium $c_6(\eta)$ are the highest coefficients $c_n(\eta)$ that we need take into account in (2). This approximation, in fact, is excellent for the calculation of $B(\theta)$ and $A(\theta)$ even for very large real values of θ. It also follows that b_i, β_i and α_i are all of the order of magnitude $\frac{1}{M^i}$. Thus the radius of convergence ϱ_θ of the power series (11) and (14) must be roughly proportional to M. From the numerical results reproduced later in this paper we obtain as a very rough estimation the value for $\varrho_\theta = \frac{M}{2}$, approximately valid for all values M and for the power series of both $B(\theta)$ and $A(\theta)$. In practice, using ten terms for example the series may be used for accurate calculation of $B(\theta)$ and $A(\theta)$ for values of $|\theta|$ approximately up to 0.4 M.

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[1] The machines actually used were *Facit*, mod. ESA, Åtvidaberg, Sweden.

Now the analytic continuation of the developments (11) and (14) is performed with the classical method using Taylor series. The method is slightly modified in taking $1/\theta$ as variable instead of θ and putting

$$y = \frac{1}{\theta}, \quad H(y) = B\left(\frac{1}{y}\right), \quad L(y) = A\left(\frac{1}{y}\right). \tag{26}$$

Then the method was found to be very useful.

The expressions of $H(y)$, $H^2(y)$ and $L(y)$ for large y-values then follow from (11) and (14). Taking now a real positive value y_0 so that $\theta_0 = 1/y_0 \ll \varrho_\theta$ (something like $0.3\,\varrho_\theta$), we calculate the values of $H(y)$[1] and $L(y)$ and their derivatives at $y = y_0$ and form the Taylor developments at this point. Taking a new point y_1 on the real positive y-axis, $\theta_1 = 1/y_1 > \theta_0$ but still a little smaller than ϱ_θ, the Taylor series at y_1 are calculated using those at y_0. We now write the new series as power series in $y - y_1$

$$H(y) = h_0 + h_1(y - y_1) + h_2(y - y_1)^2 + \cdots, \tag{27}$$

$$L(y) = l_0 + l_1(y - y_1) + l_2(y - y_1)^2 + \cdots. \tag{28}$$

The constants $h_0, h_1, \ldots, l_0, l_1, \ldots$ are calculated in the way sketched above. The choice of only one intermediate point y_0 for calculating (27) and (28) and the rough positions indicated for y_0 and y_1 are facts mainly born out by the numerical calculations which will be reproduced later. In this way, (27) and (28) imply in any case an essential extension of the knowledge of $B(\theta)$ and $A(\theta)$. In practice, of course, only a limited number of coefficients in (27) and (28) may be calculated, and this number mainly depends on the number of

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[1] It is somewhat simpler to start this calculation with the function $H^2(y)$.

b_ν, β_ν and α_ν which are initially computed. This also determines the accuracy of the values obtained for $B(\theta)$ and $A(\theta)$ for large real values of θ. This accuracy can be tested by putting $y=0$ in (27) and (28) because according to (10) and (15) the values $H(0)=1$, $L(0)=0$ are known. However, although (27) and (28) seem to show good convergence even for $y=0$, for very small $|y|$ more accurate values can be obtained by taking developments around $y=0$, $(\theta=\infty)$, and using (10) and (15). We now perform an analytic continuation of $H(y)$ and $L(y)$ to $y=0$, calculating $H(0)$, $H'(0)\dots L(0)$, $L'(0)\dots$ using (27) and (28). But before we do that (27) and (28) are completed, if needed, with terms above those actually calculated, these rest terms estimated so that we get $H(0)=1$, $L(0)=0$ exactly. This procedure is of course a little arbitrary, but in practice these correction terms are small, applied only in order to increase somewhat the accuracy of the values obtained for the derivatives at $y=0$. We finally obtain in the neighbourhood of $y=0$ $(\theta=\infty)$

$$H(y)=1-k_1 y-k_2 y^2-k_3 y^3-\cdots, \tag{29}$$

$$L(y)=m_1 y+m_2 y^2+\cdots, \tag{30}$$

where the new constants are calculated from (27), (28) as sketched above.

Now we have explicit expressions of $B(\theta)$ and $A(\theta)$ for all real positive θ-values consisting for both, of three developments: around $\theta=0$, at $\theta=\infty$ and at an intermediate point $\theta=\theta_1$. The question of the radius of convergence of the new series is best answered in detail in each special case, when numerical calculations have been performed. However, it seems in any case that the series at $\theta=0$ and $\theta=\infty$ have a common part of convergence in the θ-plane. As we also have taken an intermediate point θ_1 then for any θ-value in this interval we need only take the first few terms of a rapidly convergent development to get accurate values of $B(\theta)$ and $A(\theta)$.

As appears from (17) accuracy of the value of $B(\theta)$ is more important than of $A(\theta)$, especially for large real θ, as will be seen in detail later. Therefore most of the numerical work in this case has been devoted to $B(\theta)$. In particular, it is convenient for the evaluation of the integral (17) to have some simple approximate formulas of $B(\theta)$, called $B_s(\theta)$, as well as the exact expressions. For small $|\theta|$ we may for that purpose use (11), taking $B_s(\theta)$ as

$$B_s(\theta)=C\sqrt{\theta}\,(1-\beta_1\theta). \tag{31}$$

For small $|\theta|$ this is a good approximation, and for any real positive θ-value when this is not valid, it was found numerically that another simple expression could be used

$$B_s(\theta)=a-\frac{b}{c+\theta}. \tag{32}$$

Here, a, b, and c are constants to be determined numerically. However, if we want great accuracy, we cannot use the same values of the constants in the whole interval from a small θ-value to $\theta=\infty$. But in dividing this interval into only a few smaller intervals, and taking, for each of these, certain values of a, b, c, (32) was found to give a surprisingly good approximation not even of $B(\theta)$ but also, when differentiated, a good approximation of the first derivatives of $B(\theta)$. In particular, according to (29) for sufficiently large real θ we may take $a=1$, $b=k_1\,(>0)$ and $c=-k_2/k_1$. For the remaining intervals a is in all cases close to 1, b and especially c vary more $(b>0)$.

Using the reverse order of integration in (1) WICK [11] has outlined a procedure for calculating the functions then corresponding to $B(\theta)$ and $A(\theta)$ for real positive θ. For small θ-values he uses developments, too, but taking only some few terms of them. For intermediate θ-values he sketches a purely numerical method, giving the values of the functions only point for point instead of as an explicit expression. Finally for large θ-values WICK attacks the problem by examining directly the functions in the neighbourhood of $\theta=\infty$, and describes a method of calculating k_1, k_2 and m_1. However, the direct study of the expansions (29) and (30) around $\theta=\infty$, is difficult because no analytic formulas of the coefficients can be derived and the computation of k_2 for example is involved already. $\theta=\infty$ is therefore not a convenient starting-point for a close investigation of $B(\theta)$ and $A(\theta)$, in sharp contrast with the point $\theta=0$, from which in fact we have calculated these functions for all θ-values in this interval. The separate methods developped by WICK respectively for intermediate and large θ-values are not necessary. Besides, the method sketched here gives analytic and, except possibly for large θ, more accurate expressions of $B(\theta)$ and $A(\theta)$.

4. Evaluation of formulas for the neutron density

It is now possible to perform the integration in θ which was left in (17). We shall use the saddle-point method, applied to this problem before by WALLER [9] and by WICK [11]. The integrand in (17) consists of the product of the slowly variable function $A(\theta)$ and an exponential function, being rapidly variable compared to $A(\theta)$ provided that not both x and R are small. We may deform the path of integration L_θ in the θ-plane to pass through a saddle-point along a path of "steepest descent". Along this path in both directions the real part of the exponent is rapidly decreasing from a largest value at the saddle-point, the neighbourhood of which is the only part of L_θ that gives an essential contribution to the integral. This may then easily be evaluated by developing the integrand around this point [2]. In our case we have only one saddle-point lying always on the real positive θ-axis. Now we want to get the integral as an analytic expression in x and R and so, to begin with we must be able to express the saddle-point analytically in x and R. The method of integration, as it has been sketched above, is therefore slightly modified in the following way. Instead of the exponential function in (17) we take as the rapidly varying function that with $B(\theta)$ substituted by $B_s(\theta)$, and the difference $B(\theta)-B_s(\theta)$, being small may even when multiplied with $-R$ in the exponent be considered as a slowly variable function just as $A(\theta)$. We thus determine the saddle-point θ_s by the equation

(33) $$x-RB_s'(\theta)=0, \text{ with the root } \theta=\theta_s.$$

Thus θ_s is a function of x and R only in the combination R/x, a fact which has been stressed before [11]. $B'(\theta)$ and $B_s'(\theta)$ are real, positive, monotonically decreasing functions when θ varies along the real positive θ-axis with the limiting values 0, for $\theta \to \infty$. Then θ_s must be real, positive and monotonically increasing with R/x with $\lim \theta_s \to \infty$, for $R/x \to \infty$, ($\theta_s = 0$, for $R/x = 0$); this is seen in detail from (31) and (32). The division of the real positive θ-axis into intervals with different values of a, b, and c which was found to be necessary in order to get good approximations $B_s(\theta)$ then simply involves a similar division of the values of R/x into some few intervals.

For the evaluation of the integral (17) we should now develop the integrand around $\theta = \theta_s$ in the usual manner and then integrate, the result being a certain series development for $F_0(R, E)$. However, we must first observe that $B(\theta)$ is not regular for $\theta = 0$, thus for $R/x \to 0$ i.e. $\theta_s \to 0$ this method cannot be used. Therefore, at least for small values of R/x, (17) must first be transformed by substituting

$$\theta = \varphi^2 \quad (+\sqrt{\theta} = \varphi). \tag{34}$$

Introducing then for simplicity

$$S(\varphi) = B(\varphi^2), \quad U(\varphi) = \varphi A(\varphi^2). \tag{35}$$

(17) is transformed into

$$F_0(R, E) = \frac{Q \cdot 3 h_1}{4 \pi^2 E R} \frac{1}{i} \int\limits_{L_\varphi} d\varphi \, U(\varphi) \, e^{x\varphi^2 - R S(\varphi)}, \tag{36}$$

where the path of integration L_φ originally is the picture in the φ-plane of the path of integration L_θ in the θ-plane in (17). Now (36) may be evaluated in quite the same way as is outlined above for (17) and we observe that (34) is valid for the saddle-point φ_s in the φ-plane, i.e. $\varphi_s = +\sqrt{\theta_s}$. The explicit values of θ_s and φ_s are

$$\varphi_s = \frac{RC}{x(1+\tau)}, \text{ with } \tau = \sqrt{1 + \frac{3\beta_1 C^2 R^2}{x^2}}, \tag{37}$$

for small $\varphi_s(\theta_s)$ within the range of validity of the approximation (31) i.e. small R/x, and

$$\theta_s = \varphi_s^2 = \sqrt{\frac{bR}{x}} - c \tag{38}$$

for larger values of R/x and θ_s where (32) is valid. $\theta_s = \varphi_s^2 \leq 1/6\beta_1$ may be taken as an approximate upper limit for the use of (37), giving $R/x \leq \frac{2}{C}\sqrt{\frac{2}{3\beta_1}}$; for larger values of R/x (38) is the best approximation. (37) was derived and first used by WALLER [9], it has also been used by the present author before

[3], [4]. In the special case $R/x \to \infty$, when $b = k_1$ and $c = -k_2/k_1$ (38) has been used earlier by WICK [11].

We now form the Taylor developments of $x\varphi^2 - RS(\varphi)$ and of $U(\varphi)$ at $\varphi = \varphi_s$, taking

$$\varphi = \varphi_s + it. \tag{39}$$

All terms of the exponent, except $x\varphi_s^2 - RS(\varphi_s)$ and $-[x - RS''(\varphi_s)/2]t^2$ are then expanded and we integrate from $t = -\infty$ to $t = +\infty$. The path of integration is thus certainly not the correct path of "steepest descent" but only tangent to this at the saddle-point; however, even along the path chosen here, the real part of the exponent decreases sufficiently rapidly from the point $\varphi = \varphi_s$ and only a small central part is of importance. In this way we get from (36) an expression for $F_0(R, E)$, having the form of an infinite double series containing x, R and the functions S and U and its derivatives at $\varphi = \varphi_s$ and being valid for any value of R/x. But when (38) holds we might as well use the original integral (17) for the calculation of $F_0(R, E)$, in fact, the introduction of the functions S and U and the use of (36) is then an unnecessary complication, the derivatives of $B(\theta)$ and $A(\theta)$ being easier to calculate than those of $S(\varphi)$ and $U(\varphi)$ in this case. (17) is evaluated in the same way as (36) above, taking even here as path of integration a straight line through $\theta = \theta_s$ parallel to the imaginary axis. The expansion to be integrated is of course of identically the same type as that above in (36) and we may therefore write both expressions for $F_0\,(R, E)$ in one formula

$$\begin{aligned} F_0(R, E) = {} & \frac{Q \cdot 3h_1}{(2\sqrt{\pi})^3 RE}\, e^{x\theta_s - RB(\theta_s)}\, k \left\{ M_0 \left[1 - \frac{15 n_1^2}{16} - \frac{3 n_2}{4} + \frac{3465 n_1^4}{512} + \right.\right. \\ & \left. + \frac{945 n_1^2 n_2}{64} + \frac{105 n_2^2}{32} + \frac{105 n_1 n_3}{16} + \frac{15 n_4}{8} + \cdots \right] - M_1 \left[\frac{3 n_1}{4} - \frac{315 n_1^3}{64} - \right. \\ & \left. - \frac{105 n_1 n_2}{16} - \frac{15 n_3}{8} + \cdots \right] - M_2 \left[\frac{1}{2} - \frac{105 n_1^2}{32} - \frac{15 n_2}{8} + \cdots \right] + \\ & \left. + M_3 \left[\frac{15 n_1}{8} + \cdots \right] + M_4 \left[\frac{3}{4} + \cdots \right] + \cdots \right\}. \end{aligned} \tag{40}$$

Here, for abbreviation, we have

$$M_i = \sum_{j=0}^{i} m_{i-j} \frac{\varepsilon^j}{j!} \qquad (i = 0, 1, 2, \ldots). \tag{41}$$

In (40) and (41) we have introduced k, ε, m_i and n_i as notations for certain functions of R, x (and M) (ε is in general a small quantity). For all values of R/x we may use

$$(42)\qquad k=2\left[x-\frac{R}{2}S''(\varphi_s)\right]^{-\frac{1}{2}},\quad \varepsilon=\frac{k}{2}[2\,x\,\varphi_s-R\,S'(\varphi_s)],$$

$$m_i=\left(\frac{k}{2}\right)^i\frac{U^{(i)}(\varphi_s)}{i\,!}\ (i=0,\,1,\,2,\,\ldots),\ n_{i-2}=\left(\frac{k}{2}\right)^i\frac{R\,S^{(i)}(\varphi_s)}{i\,!}\qquad (i=3,\,4,\,\ldots),$$

obtained from (36). However, as was mentioned before, (42) is convenient to use only for small values of R/x when (37) is valid, otherwise it is simpler to take

$$(43)\qquad k=\left[-\frac{R}{2}B''(\theta_s)\right]^{-\frac{1}{2}},\quad \varepsilon=k\,[x-R\,B'(\theta_s)],$$

$$m_i=k^i\frac{A^{(i)}(\theta_s)}{i\,!}\ (i=0,\,1,\,2,\,\ldots),\ n_{i-2}=k^i\frac{R\,B^{(i)}(\theta_s)}{i\,!}\qquad (i=3,\,4,\,\ldots),$$

these values being obtained from (17) and used when (38) holds. Regarding M_i and n_i as small quantities of the i:th order of magnitude, in the expansion of $F_0(R,\,E)$ in (40) terms up to the fourth order have been calculated, and the formula therefore looks rather complicated. It is well known that, in general, a series of this type is not convergent but still can be used for practical calculation if the first terms decrease sufficiently rapidly. We may also terminate the series earlier if the first terms in (40) are small, this of course also depends on the accuracy wanted.

The formula for $F_0(R,\,E)$ given here contains a great many earlier formulas as special cases. In neglecting all terms but the first in the expansions (11), (14) and (40) we obtain the age-approximation of the neutron density. The condition for R/x in this case is, according to (37) $R/x \leqslant 1/C\sqrt{3\beta_1}$. For larger values of R/x, approximately up to this limit the formula of WALLER [9] can be used. This was obtained by taking one more term in the series (11) and (14) and using (37) and is contained in (40) by taking terms up to the second order. For still larger values of R/x more terms must be taken into account in (11) and (14) and the formula becomes more involved. For all R/x approximately up to $2\sqrt{2}/C\sqrt{3\beta_1}$, i.e. as long as (37) is valid, an earlier formula by the author [4] may be used which also contains the development of MARSHAK [6]. In the derivation of this formula only the main part, $(x\tau)^{-1/2}$, of $\frac{k}{2}$ was kept in the term $-4\,t^2/k^2$ of the exponent before the integration and in the expansion the explicit dependence on R, x and the coefficients of the series (11) and (14) was written out, taking these into account up to β_3 and α_3. It may be pointed out, however, that the way of writing the expansion of $F_0(R,\,E)$ in that paper is not strictly consistent for the largest R/x in question when (40) is somewhat better. This is on the other hand not a strictly consistent development for small R/x, but it seems unnecessary to rewrite it for this case, when some of the simpler formulas mentioned above may be used. Finally, the condition for these formulas and for the expansion (40) to be useful (when $R/x \lesssim 2\sqrt{2}/C\sqrt{3\beta_1}$) may be approximately written $x\tau \gg 1$ (say $\gtrsim 10$), obtained from (40), (42).

When R/x surpasses the above mentioned value, (38) gives the best value of the saddle-point and (43) is simplest for the terms of the expansion of $F_0(R, E)$. Then, for intermediate values of R/x (40) now contains just as before the numerical formula by WICK [11] and for $R/x \to \infty$ his asymptotic formula, obtained by taking only the term M_0 of the expansion (40). If the approximation $B_s(\theta)$ of $B(\theta)$ (32) is determined sufficiently well, we may even here, instead of the accurate but rather involved formula (40), derive a simpler approximate formula, obtained using $B_s(\theta)$ instead of $B(\theta)$ in (40) and (43). We then get

$$(44)\quad F_0(R, E) = \frac{Q \cdot 3\, h_1\, b^{1/4}}{(2\sqrt{\pi})^3\, E\, (Rx)^{3/4}}\, e^{-aR+2\sqrt{bRx}-cx} \Bigg\{ A(\theta_s) \left[1 - \frac{3}{16\sqrt{bRx}} - \frac{15}{512\, bRx} + \cdots\right] - \frac{A'(\theta_s)}{x}\left[\frac{3}{4} - \frac{15}{64\sqrt{bRx}} + \cdots\right] - \frac{A''(\theta_s)}{x}\sqrt{\frac{bR}{x}}\left[\frac{1}{4} - \frac{45}{64\sqrt{bRx}} + \cdots\right] + \frac{A'''(\theta_s)}{x^2}\sqrt{\frac{bR}{x}}\left[\frac{5}{16} + \cdots\right] + \frac{A^{\mathrm{IV}}(\theta_s)}{x^2} \cdot \frac{bR}{x}\left[\frac{1}{32} + \cdots\right] + \cdots \Bigg\}.$$

$\left|\dfrac{A^{(i)}(\theta_s)}{i!}\right|$ is decreasing for increasing i, R/x and θ_s which may be seen from (30). Then it also appears that (44) and (40) are essentially expansions in powers of $1/\sqrt{bRx}$. The condition for these expansions to be useful, for the values of R/x in question, is therefore $\sqrt{bRx} \gg 1$, (say $\gtrsim 4$). In (44) a, b and c are constant within the intervals of R/x mentioned above. The fact that b has to be altered at the endpoints of these intervals essentially determines the accuracy of (44) compared to (40) for intermediate R/x and gives then a maximum error of (44) of a few %. For very large R/x the accuracy of both (40) and (44) is rather critically dependent on the accuracy of the values of $B(\theta_s)$ and $B_s(\theta_s)$ respectively, being ultimately restricted by the limited knowledge of the values of M. In this case we must, therefore, always be content with a good value of $\log[F_0(R, E)]$. This is realised by (40) and (44) even though $B(\theta)$ has not been so accurately determined in this case which seems to be of less importance.

Finally we must discuss the validity of (40) with respect to the approximations made in deriving (17) from (1). In the first place the contribution to F_0 in (1) from all other poles of f_0 than the main pole was neglected. The reason for this approximation to be legitimate was touched a little above; more closely it has been investigated by WICK [11] and his results were also confirmed by rough numerical calculations which were made for the "h_1-pole". Secondly, the integral around the cut $+i - +i\infty$ in the ω-plane was neglected. This "branch-point" contribution is also of a type studied before both in diffusion theory [7] and slowing down theory [6]. Therefore it is not necessary to make a detailed calculation here. The essential point is that this contribution is at most the magnitude e^{-R} and is therefore neglible (cf. (44)) except for very small

R when also x is small. Summing up, the conditions given above for the expansions of $F_0(R, E)$ to be useful, are also sufficient to ensure the validity of the approximations made in deriving (40) from (1).

5. Applications to special values of *M*

Now we are going to present some numerical results for different M values and also study in general how $B(\theta)$ and $A(\theta)$ depend on M. Detailed numerical calculations have been performed for deuterium and carbon. Furthermore, for the coefficients of (11) and (14) which are the most important of the three developments of $B(\theta)$ and $A(\theta)$ and the only ones having analytical expressions in M, explicit formulas, for $M \gg 1$ are given for some of the first coefficients in form of power series in $1/M$. For completeness and comparison the corresponding values for $M = 1$ were also calculated[1], even though this has little practical interest on account of the assumption of a constant mean free path. We first present these general values for $M \gg 1$, followed in parentheses by those for $M = 1$. The constant h_1, occurring for instance in (40), has the value $h_1 = 1 - \frac{2}{3M}$.

$$(45)\quad \begin{cases} C = \sqrt{\dfrac{6}{M}}\left(1 - \dfrac{2}{3M} + \dfrac{1}{6M^2}\cdots\right) \text{ for } M \gg 1, \ (C = 1 \text{ for } M = 1), \\ \beta_1 = \dfrac{1.8}{M}\left(1 - \dfrac{2}{3M} + \dfrac{14}{45M^2}\cdots\right) \qquad \left(\dfrac{11}{30}\right), \\ \beta_2 = -\dfrac{2.870}{M^2}\left(1 - \dfrac{0.844}{M} + \dfrac{0.59}{M^2}\cdots\right) \qquad (0.0256), \\ \beta_3 = \dfrac{3.057}{M^3}\left(1 - \dfrac{0.615}{M} + \dfrac{0.65}{M^2}\cdots\right) \qquad (-0.5247), \\ \beta_4 = -\dfrac{1.82}{M^4}\left(1 - \dfrac{0.24}{M}\cdots\right) \qquad (1.238), \\ \beta_5 = \dfrac{0.72}{M^5}\cdots \qquad (-2.31). \end{cases}$$

$$(46)\quad \begin{cases} \alpha_1 = -\dfrac{3.867}{M}\left(1 - \dfrac{0.782}{M} + \dfrac{0.43}{M^2}\cdots\right) \quad \left(-\dfrac{4}{5}\right), \\ \alpha_2 = \dfrac{8.008}{M^2}\left(1 - \dfrac{0.863}{M} + \dfrac{0.50}{M^2}\cdots\right) \qquad (0.3606), \end{cases}$$

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[1] For more complete numerical results for this case cf. WICK [11].

$$(46)\quad \begin{cases} \alpha_3 = -\dfrac{9.077}{M^3}\left(1-\dfrac{0.419}{M}+\dfrac{0.43}{M^2}\cdots\right) & (0.3520), \\ \alpha_4 = \dfrac{4.95}{M^4}\left(1-\dfrac{0.55}{M}\cdots\right) & (-1.412), \\ \alpha_5 = -\dfrac{4.7}{M^5}\cdots & (3.01). \end{cases}$$

The developments with the three terms calculated in $1/M$ give good values even for deuterium, and for carbon the agreement is excellent; the remaining series give in any case even for deuterium at least the right order of magnitude. For a detailed study of the accuracy of (45) and (46) we refer to the direct calculations for deuterium and carbon which will now be quoted.

The coefficients calculated in (45) and (46) are sufficient for the knowledge of the neutron density within a wide range around the source and in any case for all values of R/x when (37) is used. However, for an accurate performance of the analytic continuation more coefficients are needed. It would be quite possible to derive general formulas in the form of power series in $1/M$ even for higher coefficients but it is rather laborious. Besides, in performing the analytic continuation of (11) and (14) and using (26) the analytic dependence of M will in any case be lost so that the coefficients of (27), (28), (29) and (30) must be determined separately for each M-value. Therefore we refrained from computing more β_n and α_n values in the general case, restricting us to deuterium and carbon.

Deuterium. We have taken $M^1 = 1.9963$, $y_0 = 3$, $\theta_0 = 1/3$ and $y_1 = \theta_1 = 1$. The values then obtained for the coefficients of the developments of $B(\theta)$ and $A(\theta)$: (11), (27), (29) and (14), (28), (30) are given in Table 1 taken in this order with the values referring to $B(\theta)$ in the three first columns.

Table 1

$C = 1.2046$

n	β_n	h_n	k_n	α_n	l_n	m_n
0	(−1)	0.7785	(−1)	(1)	0.399	(0)
1	0.63273	−0.1540	0.337	−1.2935	0.255	0.61
2	−0.47235	0.0432	−0.21	1.2769	−0.097	−0.3
3	0.29596	−0.0137	0.2	−0.9254	0.033	
4	−0.13654	0.0052		0.4430	−0.010	
5	0.03379	−0.0025		−0.0793	0.003	
6	0.00647			−0.0531		
7	−0.00700			0.0166		
8	−0.00333			0.0502		
9	0.00745			−0.0569		
10	−0.00361			0.0090		
11	−0.00185			0.0348		
12	0.00364					
13	−0.00155					
14	−0.0012					
15	0.0020					
16	−0.0007					

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[1] The values of M are obtained from BETHE, "Elementary Nuclear Theory".

In this table the last figure of the values of k_n, m_n and also of the four highest h_n and l_n has to be regarded as uncertain since it has been obtained by estimation of the rest term of a series. We have, however, possibilities of an indirect test of the values for instance by comparison of the series within a part of the θ-plane of common convergence, or for (27) and (29) by putting $y=0$. These tests give positive results. Concerning the long columns of coefficients β_n and α_n of course only the first are needed explicitly in (40), at most five or six, the others being calculated only in order to obtain the values in the other columns. As appears from these, the new series are rapidly convergent and give good values of $H(y)$ and $L(y)$, (27) and (28) for $|y-1| \lesssim 0.8$, (29) and (30) for $|y| \lesssim 0.2$. Thus in (40) it is best to use for $B(\theta_s)$ and $A(\theta_s)$, (11) and (14) for $\theta_s \lesssim 5/9$, then (27) and (28) up to $\theta_s \lesssim 5$ and in the remaining interval up to ∞ (29) and (30). These values of θ_s correspond to the following values of R/x: 2.97, 92 (and ∞), obtained from the values of a, b, and c in (32) and (38) which will now be reproduced in Table 2. This table also contains the values of certain constants d, e and f introduced in an approximate expression of $A(\theta)$, $A_s(\theta)$, similar to (32)

$$A_s(\theta) = d + \frac{e}{f+\theta}. \tag{47}$$

This was found to be a good approximation and in (44) we may use (47) for $A(\theta)$, thus getting a somewhat simpler formula. In Table 2 the two first columns give the approximate intervals of θ_s and R/x corresponding to unchanged values of $a \ldots f$ so that each row of values $a \ldots f$ is put between the approximate endpoints of the intervals within which these values ought to be used.

Table 2

θ_s	R/x	a	b	c	d	e	f
0.35	1.8						
		0.9871	0.2845	0.3637	−0.047	0.762	0.712
1.0	6.5						
		0.9971	0.3126	0.430	0	0.612	0.53
3.0	38						
		1	0.3325	0.534	0	0.612	0.53
10	330						
		1	0.337	0.62	0	0.61	0.5
∞	∞						

The values in the last row were obtained directly from k_n and m_n in Table 1 with $b=k_1$, $c=-k_2/k_1$, $e=m_1$ and $f=-m_2/m_1$, the three first of which are contained in the asymptotic formula for $R \to \infty$ of the neutron density. The other values were calculated simply by taking $B(\theta)$ and $A(\theta)$ respectively at three points in the corresponding intervals and give a maximum error at any point of about 10^{-4} and 10^{-3} respectively.

Carbon. We have $M=11.905$, $y_0=0.6$, $\theta_0=5/3$ and $y_1=0.3$, $\theta_1=10/3$. The values of the coefficients are given in Table 3.

Table 3

$C = 0.67094$

n	$\beta_n \cdot 10^n$	h_n	k_n	$\alpha_n \cdot 10^n$	l_n	m_n
0	(-1)	0.8326	(-1)	(1)	0.406	(0)
1	1.4303	$-$ 0.5406	0.49	$-$ 3.0435	1.03	1.3
2	$-$ 1.8890	0.2997	0.4	5.2583	$-$ 1.44	2
3	1.7255	0,324	-0.4	$-$ 5.2038	1.1	
4	$-$ 0.8984	$-$ 1.59		2.4295	2.0	
5	0.2097	3.2		$-$ 1.0422	-12	
6	$-$ 0.3861	$-$ 2.3		3.7191		
7	0.7172	-11		$-$ 3.324		
8	0.1526	$6 \cdot 10^1$		$-$ 6.413		
9	$-$ 1.2781	$-$ $2 \cdot 10^2$		8.301		
10	$-$ 0.2197	$6 \cdot 10^2$		16.60		
11	2.927			-25.6		
12	$-$ 0.029					
13	$-$ 6.895					
14	1.407					
15	16.62					
16	$-$ 6.89					
17	-40.49					
18	26.10					
19	98.5					
20	-89					

What was said about the values of h_n, k_n, l_n and m_n for deuterium in Table 1 is also valid here, and in particular the values of k_2, k_3 and m_2 are very rough. The developments (27) and (28) show poorer convergence at $y = 0$ in this case, than for deuterium and therefore the analytic continuation of these series to $y = 0$ was combined with graphical interpolation for the calculation of k_n and m_n. Rapidly convergent expressions of $B(\theta_s)$ and $A(\theta_s)$ in (40) are obtained if we take (11) and (14) for $\theta_s \lesssim 2.5$, then (27) and (28) up to $\theta_s \lesssim 9$ and in the remaining interval up to ∞ (29) and (30). The corresponding values of R/x are 13.1, 111 (and ∞), obtained from Table 4 which contains the values of $a \ldots f$, arranged in the same way as Table 2.

Table 4

θ_s	R/x	a	b	c	d	e	f
1	4.6						
		1.0683	1.0574	1.1947			
2	9.6						
		1.0421	0.8938	0.933	-0.187	3.761	3.01
3.7	24						
		1.0089	0.6164	0.148	-0.055	2.203	1.43
6	61						
		1.0029	0.548	-0.19	-0.004	1.48	0
13	300						
		1	0.49	-0.8	0	1.3	-2
∞	∞						

These values were calculated in the same way and give about the same accuracy for $B(\theta)$ and $A(\theta)$ as those for deuterium in Table 2. Finally as an illustration the logarithmic neutron density corresponding to $E_0 = 0.1$ Mev,

$E=1$ ev is shown graphically in Figure 1 together with curves obtained from the age-approximation and the asymptotic formula.

We have now collected enough information of the functions $B(\theta)$ and $A(\theta)$ to be able to study in general their dependence on M, at least if only the values of physical interest $M=1, 2, \ldots$ (approximately) and real positive θ-values are considered. If we write for a moment $B_M(\theta)$ and $A_M(\theta)$ (45) and (46) imply at least for small constant θ-values $B_{M_1}(\theta) > B_{M_2}(\theta)$ and $A_{M_1}(\theta) < A_{M_2}(\theta)$ for $2 \leq M_1 < M_2$. These relations also seem to hold in the whole interval for $0 < \theta < \infty$ but then only for sufficiently large values of M_1 and M_2. For large θ the relations are valid because then m_1 and k_1 increase with M; in fact, in the limiting case $M \to \infty$ for which some preliminary computations were made, taking θ/M as variable we get k_1 and m_1 proportional to M which justifies our statement, at least for large M. But they seem to be valid for a value of M as low as 5 (roughly), though we would need more numerical calculations to study this in detail. However, for the smallest M-values we have on the contrary $B_1(\theta) < B_2(\theta)$ and $A_1(\theta) > A_2(\theta)$ for constant θ, $0 < \theta < \infty$. Thus, of all physically possible functions $B_M(\theta)$ and $A_M(\theta)$ those for deuterium are the largest and the smallest respectively for small θ-values; for larger θ-values the functions in an interval $2 \leq M \leq 6$ (roughly) seem to lie close to each other and the extreme values of B_M and A_M seem to be attained within this interval.[1]

More detailed information on the dependence on M for large θ-values is obtained from the calculations for deuterium and carbon, being characteristic representatives of the case of a small and large M-values. It would perhaps be expected, that the analytic continuation would be easier for carbon than for deuterium as then the original developments (11) and (14) have much larger circles of convergence. But a comparison of Table 1 and Table 3 shows that for large θ it is just the contrary, as we have pointed out above also. For example, we get poorer values of k_n and m_n for carbon than for deuterium. These unexpected facts are best clarified if $H(y)$ and $L(y)$ are studied graphically for small y. For deuterium these functions behave gently and reach their values at $y=0$ monotonically and with no change of the sign of the curvature for decreasing y. But for carbon there is a point of inflexion in the neighbourhood of $y=0$ which causes trouble when the analytic continuation is performed beyond it. This behaviour is also seen from Table 4 which shows some characteristic differences from Table 2. The values of $a \ldots f$ vary much more for carbon than for deuterium. In particular, for carbon c (and f) gets negative for large θ_s which is the most direct indication of the existence of a point of inflexion.

What has been said above for carbon is certainly also valid in general for large M-values. In fact, the preliminary computations in the limiting case $M \to \infty$ show that these properties of $B(\theta)$ and $A(\theta)$ are then still more marked. But this also has the effect that the asymptotic formula for $R \to \infty$ of the neutron density is a much poorer approximation for intermediate R/x for large M than for small M. It is therefore also of less practical importance in the former case to perform the analytic continuation accurately as far as to $y=0$, determining nice values of the constants of the asymptotic formula.

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[1] Cf. also for k_1 some computations by MARSHAK [6].

6. Discussion of the formulas for the neutron density

We have now derived general formulas for the neutron density and studied, in general, the dependence on M of the included quantities. Therefore we are finally ready for a discussion of the neutron density and its dependence on the three parameters R, $x(E)$ and M. Then we write for (40)

$$F_0(R,E) = \frac{Q}{E x^{3/2}}\mu\, e^{-\nu R}, \tag{48}$$

where $\nu = \nu_M(R/x)$ and also, to the first approximation, $\mu = \mu_M(R/x)$. The most important of these functions is obviously ν. It seems worth mentioning first that ν very easily can be obtained graphically. In fact, let us select in the diagram of $B(\theta)$, as a function of θ, out of all straight lines having a slope x/R, that line which is tangent to $B(\theta)$. Then obviously the point of tangency has the abscissa θ_s (if the small difference $B(\theta_s)-B_s(\theta_s)$ is neglected), and the tangent cuts the ordinate axis at a point where the ordinate is precisely equal to ν. But we have also derived analytic expressions for the function $\nu = \nu_M(R/x)$. When R/x varies from 0 to ∞ ν varies monotonically from 0 to 1. At first we have $\nu = RC^2/4x$, thus ν varies rapidly with R/x and for large M is roughly proportional to $1/M$ ($\nu \approx 3R/2xM$). This holds, however, only within the small range of validity of the age-approximation or roughly when $R/x \ll 1/C\sqrt{3\beta_1}$ ($\approx M/6$ for large M; ≈ 0.6 for $M=2$), giving $\nu \ll 1/4$. For larger values of R/x in an intermediate range, ν changes to a more slowly variable function of R/x which also becomes more involved. At first ν transforms into the form derived by WALLER [9] $\nu = \dfrac{RC^2}{4x}\cdot\dfrac{4(1+2\tau)}{3(1+\tau)^2}$ (for τ cf. (37)) completed for accuracy with more terms when R/x increases [4]. Then, roughly when R/x surpasses the value $2\sqrt{2}/C\sqrt{3\beta_1}$ ($\approx M/2$ for large M) and ν attains the value 0.4 (roughly) we obtain again a simpler form of $\nu = a - 2\sqrt{\dfrac{bx}{R}} + \dfrac{cx}{R}$ which becomes more and more slowly variable with increasing R/x. It is interesting to notice now, however, that the slowest variation with R/x soon occurs for small M in contrast to the case when R/x is small. This is also connected with the fact born out by the detailed study of $B_M(\theta)$ above, that the asymptotic form of ν [11] ($a=1$, $b=k_1$, $c=-k_2/k_1$) may be used as an approximation earlier, i.e. for smaller R/x, for small M-values than for large M-values. In general, if we consider ν for any given value of R/x ($\neq 0$, $\neq \infty$) it follows let us say from the graphical method indicated above, and the discussion of $B_M(\theta)$, that ν_M decreases when M increases and M is sufficiently large (in any case for $M \geq 6$). For smaller values of M down to $M=2$ the conditions are different for small and large R/x but in any case the values of ν_M for these M lie close together. However, we always have $\nu_2 > \nu_1$ for $0 < R/x < \infty$.

Though the term $e^{-\nu R}$ in (48) is the dominating one we may also for completeness discuss the term μ. Taking $\varkappa = \left(3 - \dfrac{2}{M}\right)/(2\sqrt{\pi})^3$, this term starts with the value $\varkappa C$ (roughly proportional to $1/\sqrt{M}$ for large M) for $R/x = 0$, and is at first almost constant for very small R/x. But very soon outside the age-approximation it transforms into $\mu = \varkappa C \frac{2}{\sqrt{\tau}(1+\tau)}$ (roughly) [9], [4] and now decreases more and more rapidly. When R/x has increased to the value $2\sqrt{2}\, C \sqrt{3\beta_1}$ (roughly), we have $\mu = \varkappa b^{1/4} (x/R)^{3/4} A(\theta_s)$ (approximately), varying at first roughly as x/R but for increasing R/x transforming into the asymptotic value $\mu = \varkappa m_1 (k_1)^{-1/4} (x/R)^{5/4}$ [11]. This quantity is increasing with M for large M so that the dependence on M is reversed compared to the value $\mu_M(0)$. But of course this fact is of little importance in (48) compared to the influence of the exponential.

This discussion of ν and μ also gives the full picture of the neutron density as function of R, x and M with the restriction of course that the conditions must be fulfilled, for (40) to be valid, i.e. roughly $x \geq 10$ for small R/x and $\sqrt{bRx} \geq 4$ for large R/x, these are also essential for the discussion of μ above. We may study for instance, for every value of M, the space distribution of neutrons of constant energy, i.e. constant x. However, it is only for $x \geq 10$, approximately, that we get the complete space distribution. For higher neutron energies, i.e. smaller x, we have a small dominion around the source, extending with decreasing x where our formulas become uncertain but outside which we still have the complete space distribution. This may be described for intermediate R already as a quasi-exponential decay with a decay-length >1 slowly decreasing towards the limit 1 $(=\lambda)$.

Let us then study the energy distribution of neutrons at a certain distance R. The dependence of x for the factors μ and $e^{-\nu R}$ is then described by the discussion above. The factor left in (48) is decreasing with increasing E except when E is close to E_0 but otherwise all factors co-operate to make the energy distribution function decreasing with increasing E. This also makes the restriction, for the complete space distribution to slow neutrons pointed above, less important. When both x and R are small, source neutrons and neutrons having suffered only some few impacts are also of importance and our formulas are then insufficient. As an illustration of what this restriction implies for the energy distribution, let us take for carbon $R = 20$. Then it seems possible to use the formulas roughly for $x > 1$ which indeed is a very modest restriction. In general, at distances of this magnitude, still small enough to be of practical interest, our formulas give almost the complete energy distribution.

Finally, taking constant values of R and x, we may study how the neutron density depends on M. Taking first $R = 0$ and constant values of E_0 and E, F_0 increases rather much from $M = 1$ to $M = 2$, varies then slowly for the next M-values (maximum for $M = 4$) and decreases then roughly as $1/\sqrt{M}$. But for increasing R this picture is soon reversed with a slow variation and a minimum in the interval $2 \leq M \leq 6$ and then F_0 increases with M for larger M-values and also when we pass to $M = 1$. However, in making this comparison between different moderators it is important to remember that in general we have different length units for them, these being from the beginning chosen as the mean free paths λ_M of the neutrons. Thus we have for each M $R = r/\lambda_M$, if r is the distance from the source measured in ordinary units. To make this comparison complete we must therefore finally apply a spherical dilation of the

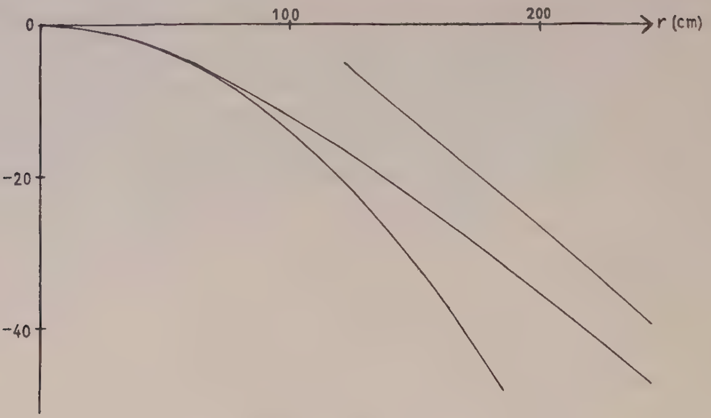

Fig. 1. Log F_0 + constant, calculated for carbon from:

$\left\{\begin{array}{l}\text{formula (40) (middle curve)}\\ \text{age-approximation (lower curve)}\\ \text{asymptotic formula (upper curve, very rough)}\end{array}\right.$ $E_0=0.1$ Mev, $E=1$ ev, $x=11.51$, $\lambda=2.65$ cm.

R-space for each M, the scale of length being λ_M to 1, and then we obtain $F_0(r, E)$. A full discussion of the dependence on M for the neutron density should therefore also require a consideration of the variation of λ_M with M which, however, is beyond the scope of this paper. But it is worth mentioning that this has a particularly large effect for hydrogen (in water for example) with its very small mean free path (E_0 small), making it a most effective element for purposes of slowing down, much better than say deuterium, in spite of the fact, pointed out above, that the slowing down per mean free path is better in deuterium.

Finally concerning the practical applicability of our formulas it must be stressed that the assumption of constant mean free path for energies in the interval $E-E_0$ is justified only for sufficiently small E_0. For carbon this holds for $E_0 \lesssim 0.1$ Mev[1], with $\lambda = 2.65$ cm (taking the carbon density as 1.60 g/cm^3). For deuterium [6] the approximation is good for $E_0 \lesssim 0.7$ Mev, the relative variation of λ then being less than 5 % (roughly). An extension to the more important case of heavy water will be given in a later paper. For hydrogen and water [6] the approximation is justified only when E_0 is at most of the order 0.01 Mev and the formulas therefore seem to be of very little practical use in this case.

For higher values of E_0 we may try to use in our formulas some mean value $\bar{\lambda}$ of the actual values of λ within the slowing down energy interval. Then of course the formulas can not be expected to give the same accuracy as when E_0 is small. However, in chosing $\bar{\lambda}$ judiciously we may still hope to get useful formulas, at least qualitatively, for moderate values of E_0. Considering λ as a function of x $\lambda = \lambda(x)$, age-theory [6] gives us the information that we should choose $\bar{\lambda}$ as

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[1] Cf. [3] and [6], where also further references to measurements of λ are given.

$$\bar{\lambda} = \sqrt{\frac{1}{x}\int_0^x \lambda^2(\xi)\, d\xi}, \tag{49}$$

for neutrons of energy $E = E_0 e^{-x}$. When the conditions given above, for the age-approximation to be valid, are fulfilled, i.e. small R/x and large x (49) must be a good value of $\bar{\lambda}$ in our formulas. Supposing now the most frequently occurring case of a mean free path which decreases with decreasing E, varying between $\lambda_{\max}$ and $\lambda_{\min}$, (49) gives a value of $\bar{\lambda}$ which is not much larger than $\lambda_{\min}$ (if not $\lambda_{\max}/\lambda_{\min}$ is too large) and varying slowly with x. Now one must expect, for physical reasons, that the fast neutrons corresponding to $\lambda = \lambda_{\max}$ increase in importance with increasing values of R (and E); mathematically this has also been shown by WICK [11] in his formula of the asymptotic neutron density for this case. Therefore, (49) gives too small a value of $\bar{\lambda}$ for increasing R (and decreasing x). In fact, for $R \to \infty$ we must have $\nu R = \nu r/\bar{\lambda} \to r/\lambda_{\max}$[1] [11]. Thus, in order to make our formulas more useful, instead of taking a mean value $\bar{\lambda}$, constant for all r and x, we start with the value (49), variable with x, for $r = 0$ and then for each x-value we let $\bar{\lambda}$ slowly increase with r towards the limit $\lambda_{\max}$ so that $\nu/\bar{\lambda}$ is monotonically increasing towards $1/\lambda_{\max}$ and $r\nu/\bar{\lambda} \to \to r/\lambda_{\max}$. With that prescription our formulas are useful for small R/x and also give the right main term of the logarithmic neutron density for $r \to \infty$ and therefore it does not seem too bold to hope that they will be at least qualitatively applicable for all distances for a moderate variation of λ. For intermediate R/x the slow increase with R/x of ν in (48) was stressed above. For the coefficient of $-r$ in the exponent $\nu/\bar{\lambda}$ the slow increase of $\bar{\lambda}$ means that this factor will be still more slowly increasing. This makes the quasi-exponential behaviour of the neutron density for a constant value of x (in disregarding the factor $1/r$ roughly) which was pointed out before, still more prominent for intermediate values of R/x; the decay-length is now larger than $\lambda_{\max}$ and slowly decreasing towards $\lambda_{\max}$ (cf. also BOTHE [1]). This behaviour is also born out by experiments, by JENSEN and RITTER among others [5].

Though it seems possible in this way to reduce the error introduced by the assumption of a constant mean free path it is in general necessary for high values of E_0 to take into account the energy dependence from the start if we want accurate formulas. Some important results for this case are due to WICK [11]. In a previous paper the present author derived formulas of the neutron density at small distances from the source, assuming a special form of energy dependence for λ [3], cf. also [4]. In a later paper this problem will be taken up again and more general formulas given in order to complete the present paper.

In conclusion we point out the slight modifications necessary for taking absorption into account, assuming the scattering mean free path λ_s, as well as the total mean free path λ, to be constant. It implies a multiplication of q and all $c_n(\eta)$ in (2), (4) with the constant $\varkappa = \lambda/\lambda_s$ $(0 < \varkappa < 1)$ [8], [9]. The main pole ω_0 is then connected with the root $\eta_0 < 1$ of the equation $h_0(\eta) = 1 - \varkappa c_0(\eta) = 0$. η_0 may be determined numerically for a given value of $\varkappa$. Taking $\theta = \eta - \eta_0$ instead of (8), the functions $\omega_0(\theta)$, $B(\theta)$ and $A(\theta)$ are defined and may be determined exactly as before, with the difference only that the constants h_n^i now refer to the values of the functions $h_n(\eta) = 1 - \varkappa c_n(\eta)$ and their derivatives at $\eta = \eta_0$. Substituting only $(E_0/E)^{\eta_0}$ for the factor E_0/E and multiplying with $\varkappa$, the neutron density will have the same form as (40) and (48), with the same general properties of the functions μ and ν. For any value of $R/x \neq 0$, ν, in particular, will be increased compared to the case treated before ($\varkappa = 1$), and the neutron density, of course, is reduced for all R and x.

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[1] More accurately: $r/\lambda_{\max} - x\,\theta_m$, where θ_m is the root of the equation $B(\theta) = \lambda_{\min}/\lambda_{\max}$. At the same time also μ has to be corrected [11], HOLTE (to be published in Ark. Fys.).

Appendix A

Proof of (16). Introducing for simplicity a new variable

$$s = \omega^2, \quad s_0(\theta) = \omega_0^2(\theta), \tag{50}$$

we write

$$\operatorname{Res}\{f_0(\theta, \omega)\}_{\omega = \omega_0(\theta)} = \frac{1}{2\,\omega_0(\theta)} \lim_{s \to s_0(\theta)} \frac{s - s_0(\theta)}{f_0^{-1}(\theta, s)}. \tag{51}$$

For small values of θ and s we have

$$f_0^{-1}(\theta, s) = q^{-1}\left\{\theta\, h_0' + \frac{\theta^2}{2}\, h_0'' + \cdots + \frac{s}{3\,h_1}(1 + \cdots)\right\}, \tag{52}$$

and, remembering the value of $C^2 = 3\,h_0' h_1$, from (9) and (11)

$$s_0(\theta) = -3\,h_1\,\theta\,(h_0' - h_0' b_1\,\theta - h_0' b_2\,\theta^2 - \cdots). \tag{53}$$

The functions (52) and (53) also depend of the constants h_n^i. If, for a moment, we substitute for one of these constants, say h_0', a variable parameter u and call the new functions simply $f_0^{-1}(\theta, s, u)$ and $s_0(\theta, u)$ we may write for (51)

$$\frac{1}{2\,\omega_0(\theta)} \lim_{s \to s_0(\theta)} \lim_{u \to h_0'} \frac{s - s_0(\theta, u)}{f_0^{-1}(\theta, s, u)} = \frac{1}{2\,\omega_0(\theta)} \left\{ \frac{-\dfrac{\partial}{\partial u}\, s_0(\theta, u)}{\dfrac{\partial}{\partial u} f_0^{-1}[\theta, s_0(\theta), u]} \right\}_{u = h_0'}. \tag{54}$$

The partial derivative in the denominator is simply $q^{-1}\,\theta$ (thanks to the choice of $u = h_0'$, more general we could have chosen $u = h_0^i$ and obtained the same simplicity) and the numerator is $3 h_1\,\theta \left\{1 - \theta \dfrac{\partial}{\partial h_0'}(h_0' b_1) - \theta^2 \dfrac{\partial}{\partial h_0'}(h_0' b_2) - \cdots\right\}$ which, in comparing with (13) and (14) proves (16).

Appendix B

Numerical calculation of the constants h_n^i for $M > 1$. For simplicity we introduce constants c_n^i deriving from $c_n(\eta)$ in the same way as h_n^i from $h_n(\eta)$ (12), thus

$$h_n = 1 - c_n, \quad h_n^i = -c_n^i \quad (i \geq 1). \tag{55}$$

Further we introduce two new constants

$$\gamma = \sqrt{M^2 - 1}, \quad \delta = \log \frac{M+1}{M-1}. \tag{56}$$

From (5) we then get

$$e^{-\delta\eta} \frac{d^i}{d\eta^i} [e^{\delta\eta} c_n(\eta)] = \frac{(M+1)^2}{2M} \int_{-1}^{+1} P_n(\mu) \cdot \\ \cdot \left\{ 2 \log \left[\sqrt{\frac{M+1}{M-1}} G(\mu) \right] \right\}^i \frac{[G(\mu)]^{2\eta}}{\sqrt{\mu^2 + \gamma^2}} d\mu. \tag{57}$$

Using the Leibnitz formula and putting $\eta = 1$ we obtain

$$\sum_{k=0}^{i} \binom{i}{k} \delta^k c_n^{i-k} = \frac{2^{i-1}\gamma}{M} \int_{-1}^{+1} P_n(\mu) \left[D\left(\frac{\mu}{\gamma}\right) \right]^i E\left(\frac{\mu}{\gamma}\right) d\mu. \tag{58}$$

Here for abbreviation two functions D and E have been introduced, having the simple form

$$D(t) = \log\left(\sqrt{1 + t^2} + t\right), \tag{59}$$

$$E(t) = 2t + \frac{1 + 2t^2}{\sqrt{1 + t^2}}. \tag{60}$$

This easily follows frow (5), (6), (56) and (57).

(58) is a general formula for a successive determination of c_n^i suitable for numerical work. In fact, the integrals in (58) are easily calculated by developing D and E

$$D(t) = \sum_{\nu=0}^{\infty} \binom{-\frac{1}{2}}{\nu} \frac{t^{2\nu+1}}{2\nu+1}, \tag{61}$$

$$E(t) = 2t + \sum_{\nu=0}^{\infty} \binom{\frac{1}{2}}{\nu} (2\nu + 1) t^{2\nu}. \tag{62}$$

Thus $tD(t)$ and $E(t)-2t$ are functions of t^2 and in (58) $t^2=\frac{\mu^2}{\gamma^2}$ has even for the smallest M-value in question $M=2$ a small value $\leq 1/3$. First we have to calculate the terms of the series development of $D^i(t)\,E(t)=2tD^i(t)+ +[E(t)-2t]\,D^i(t)$, written as the sum of an even and an odd function. But even for the application to $M=2$ it is only necessary to compute the first few terms of these developments because of the rapid convergence within the interval of integration in (58). Besides, for increasing values of i (n small) the integral becomes very small compared to the terms of the left member of (58) so that in any case a calculation of the developments up to $i\leq 8$ is sufficient. In substituting these developments in (58) we get integrals of the type

$$I_{nk}=\int_{-1}^{+1}P_n(\mu)\,\mu^k\,d\mu \quad (k \text{ positive integer}). \tag{63}$$

The computations are facilitated by the facts that I_{nk} vanishes except when $k-n$ is a positive even integer or zero, and then we get from the general formula of I_{nk}[1]

$$I_{0k}=\frac{2}{k+1},\quad I_{1k}=\frac{2}{k+2},\quad \text{and } I_{n+2,k}=I_{nk}\frac{k-n}{k+n+3} \qquad (n=0,1,2,\ldots). \tag{64}$$

The division of the function D^iE into the sum of an even and an odd function made above is now useful: in fact, when $i-n$ is odd only the part $2\frac{\mu}{\gamma}D^i\left(\frac{\mu}{\gamma}\right)$ contributes to (58) and when $i-n$ even only the other part. The resulting development in $1/\gamma^2$ for the integral begins in the former case with a term in $1/\gamma^{i+1}$ if $i+1\geq n$ and $1/\gamma^n$ if $i+1\leq n$ except of course for $i=0$ when the integral vanishes for $n\geq 3$, i.e. we have $c_3=c_5=c_7\ldots=0$. For $i-n$ even the largest term of the integral is of the order $1/\gamma^i$ if $i\geq n$ and $1/\gamma^n$ if $i\leq n$. As δ and $1/\gamma$ are of the order $1/M$ it follows from (58) that c^i_n is of the order $1/M^i$ for $i\geq n$, and $1/M^n$ for $i\leq n$ (with the exception above). From (21), (22), (24) and (25) it then follows that a_{ni}, d_{ni}, b_i, β_i and α_i all are of the order $1/M^i$. Thus for $i<n$ c^i_n loses importance, particularly as for increasing n (63) also decreases. This justifies the approximation $c_n(\eta)=0$ for sufficiently large n and it is easy to study in detail its validity in each special case.

Summary

The stationary space energy distribution of neutrons in an infinite, homogeneous slowing-down medium, containing nuclei of mass M (in units of the neutron mass), has been studied. The neutrons are supposed to be emitted isotropically with constant energy E_0 from a point source. The mean free path λ

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[1] Cf. for instance WHITTAKER and WATSON, "Modern Analysis" § 15, 211.

is assumed to be constant for energies below E_0. For neutrons of energy E at the distance R from the source (putting $\lambda = 1$) the number of collisions per unit volume, per unit energy range is

$$F_0(R, E) = \frac{Q}{E x^{3/2}} \mu e^{-\nu R},$$

where Q is the number of neutrons emitted per unit time and $x = \log E_0/E$. ν and to the first approximation also μ are functions of R/x only. Analytic expressions have been calculated for these functions. When R/x varies from 0 to ∞, ν increases monotonically from 0 to 1, at first for very small R/x proportionally with R/x, for intermediate values of R/x more slowly. In particular, for $R/x \gtrsim M/2$ (for large M-values; larger than this value for small M), roughly corresponding to $\nu \gtrsim 0.4$, in dividing these values of R/x, up to ∞, into some few intervals we have $\nu = a - 2\sqrt{\frac{bx}{R} + \frac{cx}{R}}$ where a, b and c are constant within each of these intervals (a close to 1). μ is approximately constant for very small R/x, and then decreasing with increasing R/x, for intermediate values of R/x, roughly as x/R, and, for very large R/x, as $(x/R)^{5/4}$. The formula is valid at any distance R for neutrons considerably slowed down ($x \gg 1$), but for increasing R also for higher neutron energies ($Rx \gg 1$). The assumption of constant λ is a good approximation in general only for $E_0 \lesssim 0.1$ Mev (roughly) but by chosing a suitable mean value for λ the formula may be applied even for higher E_0, at least qualitatively. Applications have been made to deuterium and carbon.

The problem of the space energy distribution of neutrons has been suggested to me by Professor IVAR WALLER, and I wish to express my gratitude to him for continued interest and encouragement and for valuable criticism during the work.

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REFERENCES. [1] **Bothe,** Z. Phys. *122,* 648, 1944. — [2] **Brillouin,** Ann. l'Ec. Norm. *33,* 3, 1916. — [3] **Holte,** Ark. Mat. Astr. Fys., *35 A,* N:o 2, 1947. — [4] ——, Ark. Mat. Astr. Fys., *35 A,* N:o 36, 1948. — [5] **Jensen** und **Ritter,** Z. Naturforsch. *2 a,* 376, 1947. — [6] **Marshak,** Rev. Mod. Phys. *19,* 185, 1947. — [7] **Marshak, Brooks** and **Hurwitz, Jr.,** Nucleonics, *4,* N:o 5, 10, 1949. — [8] **Waller,** Ark. Mat. Astr. Fys., *34 A,* N:o 3, 1946. — [9] ——, Ark. Mat. Astr. Fys., *34 A,* N:o 5, 1946. — [10] **Verde** and **Wick,** Phys. Rev. *71,* 852, 1947. — [11] **Wick,** Phys. Rev. *75,* 738, 1949.

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